Asymptotic rank bounds: a numerical census

Kisun Lee (Clemson University) - kisunl@clemson.edu

Joint Mathematics Meeting 2026

AMS Special Session on Numerical Algebraic Geometry and Its Applications

My first MRC

MATHEMATICS
RESEARCH COMMUNITIES

AMS MRC 2021

Combinatorial Applications of Computational Geometry and Algebraic Topology

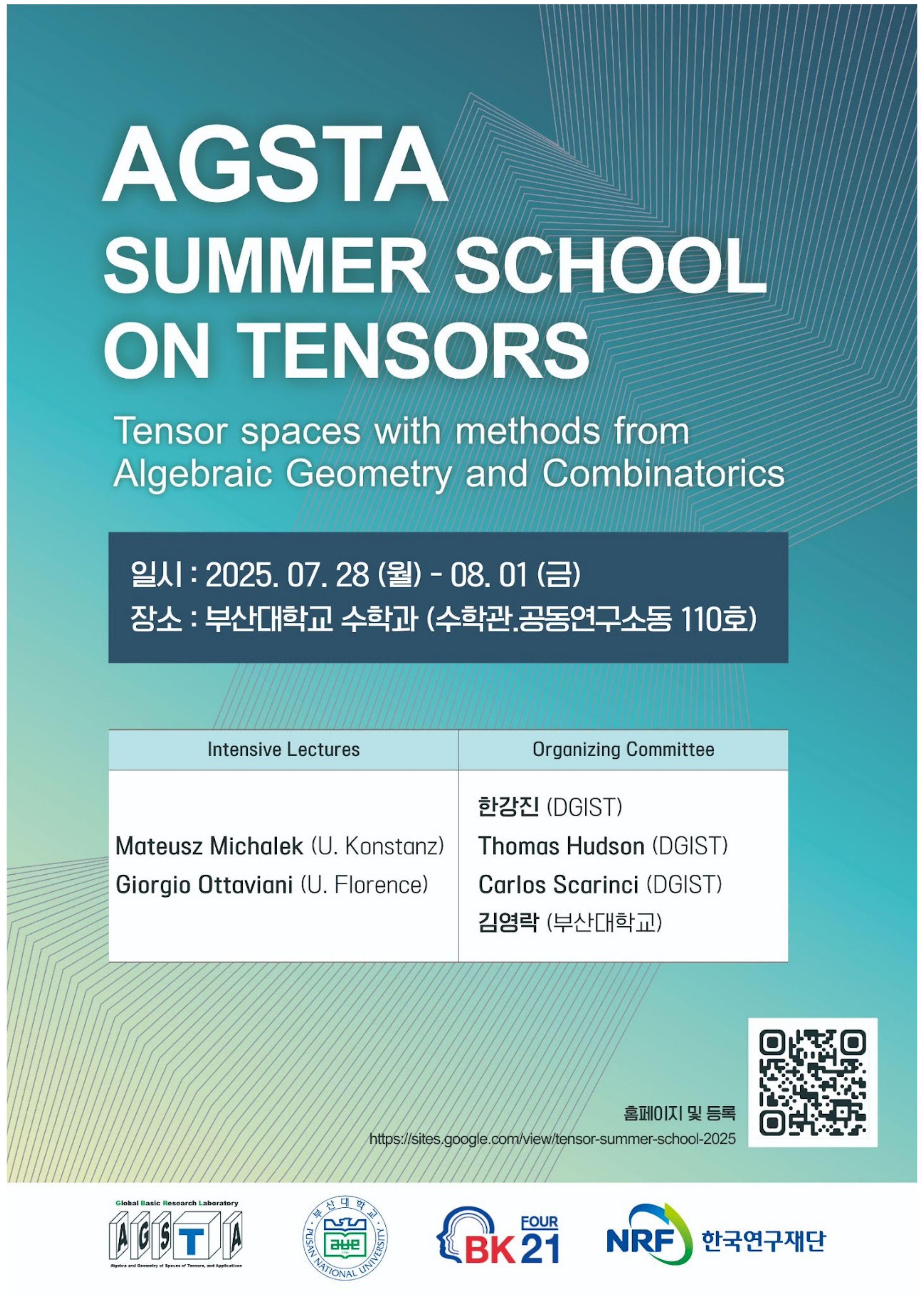
My best MRC

MATHEMATICS RESEARCH COMMUNITIES

AMS MRC 2025

Real Numerical Algebraic Geometry

Acknowledgement



Project Initiation

- **AGSTA Summer School on Tensors 2025** (Busan, Korea)
- **Special Thanks to Mateusz Michałek**

Representations of varieties

parametric vs. implicit

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times3} \mid \text{rank}(M) \le 2\} =: \sigma_2(\mathbb{C}^{3\times3})$$

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (definining equations)**

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defining equations)**
Formulation		

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (definining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	

Representations of varieties
parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times3})$$

	Parametric (sum of rank 1)	**Implicit (definining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (definining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling		

Representations of varieties
parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defininining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (definining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defininig equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)		

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (definining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	Easy (just check if $\det(M) = 0$)

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	Easy (just check if $\det(M) = 0$) $\Rightarrow$ (dim, degree, witness set)

Representations of varieties

parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defining equations)**
Formulation	$M = u_1v_1^\top + u_2v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	Easy (just check if $\det(M) = 0$) $\Rightarrow$ (dim, degree, witness set)
Numerical AG friendly?		

Representations of varieties
parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	Easy (just check if $\det(M) = 0$) $\Rightarrow$ (dim, degree, witness set)
Numerical AG friendly?		**Yes!** (homotopy continuation, witness set, ...)

Representations of varieties
parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times3})$$

	Parametric (sum of rank 1)	**Implicit (defininig equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	Easy (just check if $\det(M) = 0$) $\Rightarrow$ (dim, degree, witness set)
Numerical AG friendly?	Maybe…?	**Yes!** (homotopy continuation, witness set, …)

Representations of varieties
parametric vs. implicit

Example: Two ways to describe 3×3 matrices of rank at most 2.

$$\{M \in \mathbb{C}^{3\times 3} \mid \text{rank}(M) \leq 2\} =: \sigma_2(\mathbb{C}^{3\times 3})$$

	Parametric (sum of rank 1)	**Implicit (defining equations)**
Formulation	$M = u_1 v_1^\top + u_2 v_2^\top$	$\det(M) = 0$ (i.e., $F(x) = 0$)
Sampling	Easy (just choose random u_i and v_i)	Hard (hard to find defining eqs and solve them)
Membership test (checking if $M \in \sigma_2$)	Hard (Can you tell the rank just by looking at it?)	Easy (just check if $\det(M) = 0$) $\Rightarrow$ (dim, degree, witness set)
Numerical AG friendly?	Maybe…?	**Yes!** (homotopy continuation, witness set, …)

Numerical implicitization

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

- If we had defining equations,

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

- If we had defining equations,
  1. Intersect with a generic linear space L of complementary dimension.
  2. Compute points in $X \cap L$ (**Standard witness set**)

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

- If we had defining equations,
    1. Intersect with a generic linear space L of complementary dimension.
    2. Compute points in $X \cap L$ (**Standard witness set**)
- Instead, we have a parametrization (instead of defining equations):

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

- If we had defining equations,
    1. Intersect with a generic linear space L of complementary dimension.
    2. Compute points in $X \cap L$ (**Standard witness set**)
- Instead, we have a parametrization (instead of defining equations):

Example:

$$\begin{array}{rccc} \Phi: & (\mathbb{C}^3 \times \mathbb{C}^3)^2 & \longrightarrow & \mathbb{C}^{3\times 3} \\ & (u_1, u_2, v_1, v_2) & \longmapsto & M = u_1 v_1^\top + u_2 v_2^\top \end{array}$$

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

- If we had defining equations,
    1. Intersect with a generic linear space L of complementary dimension.
    2. Compute points in $X \cap L$ (**Standard witness set**)
- Instead, we have a parametrization (instead of defining equations):

Example:

$$\begin{array}{cccc} \Phi: & (\mathbb{C}^3 \times \mathbb{C}^3)^2 & \longrightarrow & \mathbb{C}^{3\times 3} \\ & (u_1, u_2, v_1, v_2) & \longmapsto & M = u_1 v_1^\top + u_2 v_2^\top \end{array}$$

1. **Pull back** L to the domain space and solve $\Phi(u_1, u_2, v_1, v_2) \in L$

$$\langle A_i, \Phi(u_1, u_2, v_1, v_2) \rangle = b_i, \quad i = 1, \ldots, \dim(X) \quad \text{where } A_i \text{ and } b_i \text{ define } L$$

Numerical implicitization

A pseudowitness set **(Hauenstein-Sommese 2010, 2013)**

Goal: Extract geometric info (degree, dimension) without equations.

- If we had defining equations,
  1. Intersect with a generic linear space L of complementary dimension.
  2. Compute points in $X \cap L$ (**Standard witness set**)
- Instead, we have a parametrization (instead of defining equations):

  Example:

$$\begin{array}{rccc} \Phi: & (\mathbb{C}^3 \times \mathbb{C}^3)^2 & \longrightarrow & \mathbb{C}^{3\times 3} \\ & (u_1, u_2, v_1, v_2) & \longmapsto & M = u_1 v_1^\top + u_2 v_2^\top \end{array}$$

  1. **Pull back** L to the domain space and solve $\Phi(u_1, u_2, v_1, v_2) \in L$

$$\langle A_i, \Phi(u_1, u_2, v_1, v_2)\rangle = b_i, \quad i = 1, \ldots, \dim(X) \quad \text{where } A_i \text{ and } b_i \text{ define } L$$

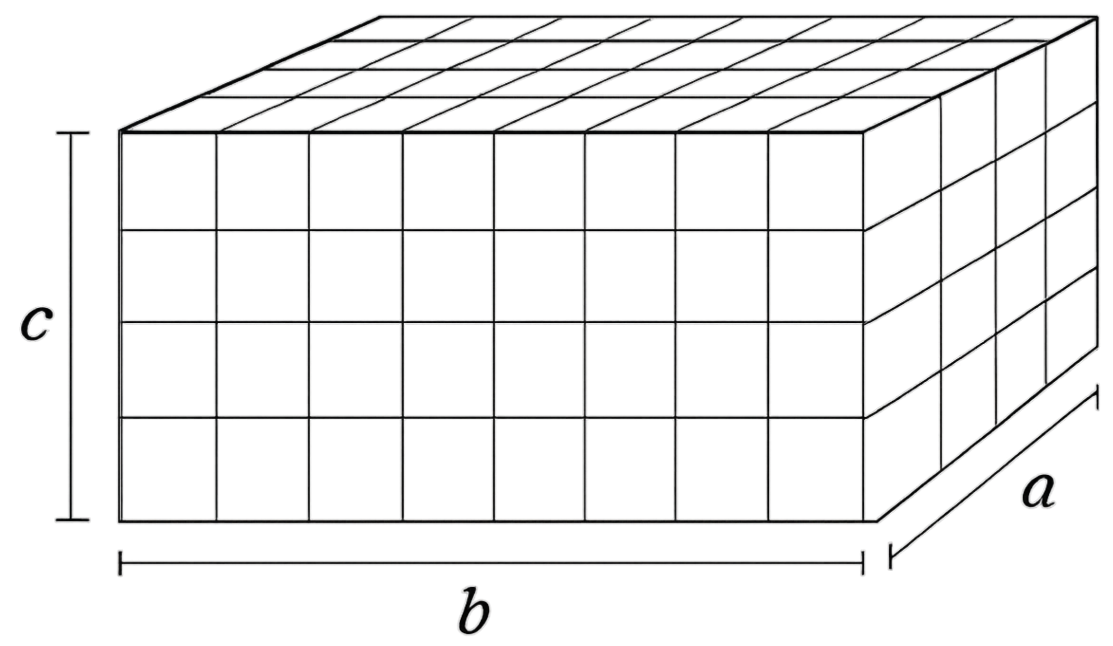
  2. Considering A_i and b_i as parameters, we find solutions using monodromy. (**Pseudowitness set**)

Asymptotic rank

Asymptotic rank

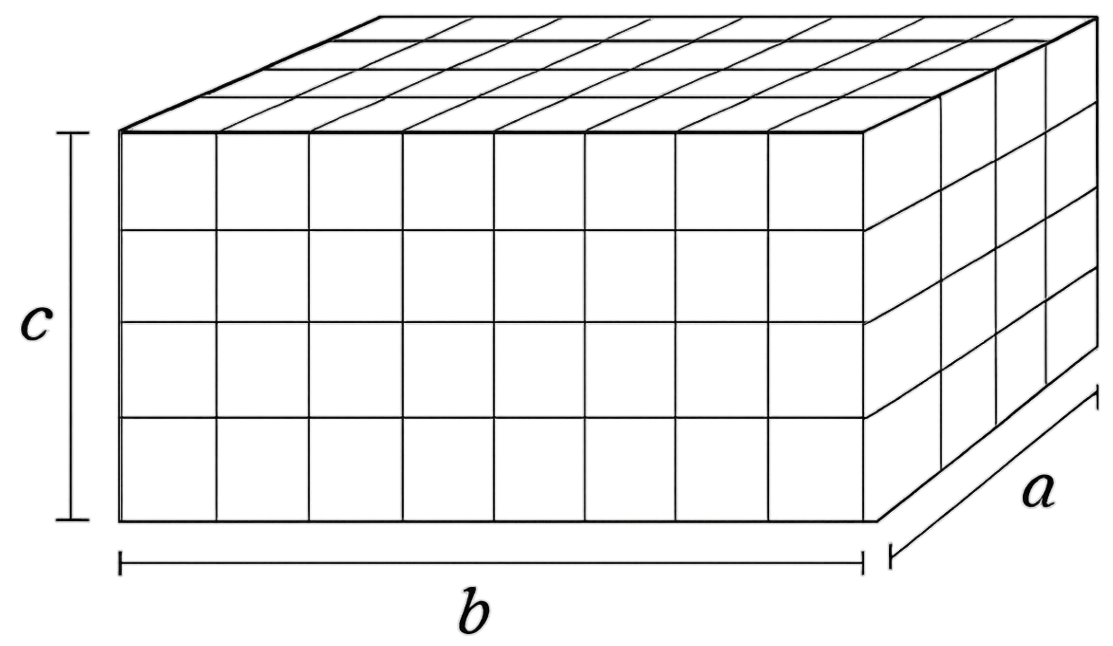
Setup: Consider a 3-way tensor space $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$

Asymptotic rank



Setup: Consider a 3-way tensor space $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ (i.e., a space of "boxes")

Asymptotic rank

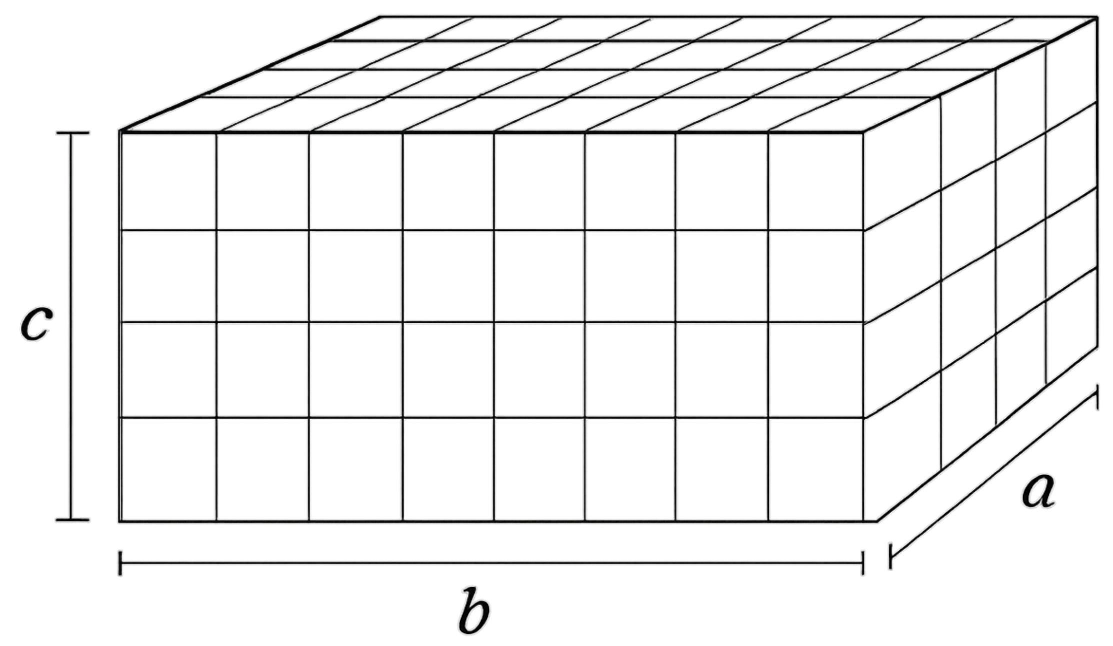


Setup: Consider a 3-way tensor space $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ (i.e., a space of "boxes")

$$\begin{aligned} \Phi_r : \quad & (\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}^c)^r && \longrightarrow && \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c \\ & (u_1, \ldots, u_r, v_1, \ldots, v_r, w_1, \ldots, w_r) && \longmapsto && T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \end{aligned}$$

Parametrization of $\sigma_r(V)$ (rank at most r tensors)

Asymptotic rank



Setup: Consider a 3-way tensor space $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ (i.e., a space of "boxes")

$$\begin{array}{cccc} \Phi_r : & (\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}^c)^r & \longrightarrow & \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c \\ & (u_1, \ldots, u_r, v_1, \ldots, v_r, w_1, \ldots, w_r) & \longmapsto & T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \end{array}$$

Parametrization of $\sigma_r(V)$ (rank at most r tensors)

Def For a tensor T, the **asymptotic rank** of T measures the growth rate of tensor powers:

$$\tilde{R}(T) := \lim_{q \to \infty} (\mathrm{rank}(T^{\otimes q}))^{\frac{1}{q}}$$

Strassen’s asymptotic rank conjecture

Strassen's asymptotic rank conjecture

Strassen's asymptotic rank conjecture:

For a concise and tight tensor $T \in V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$,

$$\tilde{R}(T) := \lim_{q \to \infty} (\text{rank}(T^{\otimes q}))^{\frac{1}{q}} = \max\{a, b, c\}$$

Strassen's asymptotic rank conjecture

Strassen's asymptotic rank conjecture:

For a concise and tight tensor $T \in V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$,

$$\tilde{R}(T) := \lim_{q \to \infty} (\mathrm{rank}(T^{\otimes q}))^{\frac{1}{q}} = \max\{a, b, c\}$$

- **concise:** $a \leq b \leq c < ab$ (injectivity of flattenings)
- **tight:** the support satisfies conservation laws (generic border rank is large)

Strassen's asymptotic rank conjecture

Strassen's asymptotic rank conjecture:

For a concise and tight tensor $T \in V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$,

$$\tilde{R}(T) := \lim_{q \to \infty} (\mathrm{rank}(T^{\otimes q}))^{\frac{1}{q}} = \max\{a, b, c\}$$

- **concise:** $a \leq b \leq c < ab$ (injectivity of flattenings)
- **tight:** the support satisfies conservation laws (generic border rank is large)

For a matrix M, the rank is multiplicative: $\tilde{R}(M) = \lim_{q \to \infty} ((\mathrm{rank}(M))^q)^{\frac{1}{q}} = \mathrm{rank}(M)$

Strassen's asymptotic rank conjecture

Strassen's asymptotic rank conjecture:

For a concise and tight tensor $T \in V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$,

$$\tilde{R}(T) := \lim_{q \to \infty} (\mathrm{rank}(T^{\otimes q}))^{\frac{1}{q}} = \max\{a, b, c\}$$

- **concise:** $a \leq b \leq c < ab$ (injectivity of flattenings)
- **tight:** the support satisfies conservation laws (generic border rank is large)

For a matrix M, the rank is multiplicative: $\tilde{R}(M) = \lim_{q \to \infty} ((\mathrm{rank}(M))^q)^{\frac{1}{q}} = \mathrm{rank}(M)$

For a tensor T, the tensor rank is sub-multiplicative: $\mathrm{rank}(T^{\otimes q}) \leq (\mathrm{rank}(T))^q$

(the asymptotic rank can be large, but the conjecture predicts a collapse to the trivial bound)

Toward Strassen's asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Toward Strassen’s asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound).

Toward Strassen’s asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

Toward Strassen's asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

1. Use the secant variety $\sigma_{18}(V)$ with the parametrization: $\Phi_{18} : (u_i, v_i, w_i) \mapsto T = \sum^{18} u_i \otimes v_i \otimes w_i$

Toward Strassen's asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

1. Use the secant variety $\sigma_{18}(V)$ with the parametrization: $\Phi_{18} : (u_i, v_i, w_i) \mapsto T = \sum^{18} u_i \otimes v_i \otimes w_i$
2. By **numerical implicitization**, we know that $\sigma_{18}(V)$ is codim 1 (a hypersurface) of degree ≥ 187000 (i.e., $|\sigma_{18}(V) \cap L| \geq 187000$) **(Hauenstein-Ikenmeyer-Landsberg 2013)**

Toward Strassen’s asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

1. Use the secant variety $\sigma_{18}(V)$ with the parametrization: $\Phi_{18} : (u_i, v_i, w_i) \mapsto T = \sum^{18} u_i \otimes v_i \otimes w_i$
2. By **numerical implicitization**, we know that $\sigma_{18}(V)$ is codim 1 (a hypersurface) of degree ≥ 187000 (i.e., $|\,\sigma_{18}(V) \cap L\,| \geq 187000$) **(Hauenstein-Ikenmeyer-Landsberg 2013)**
3. **There is no degree 186999 homogeneous polynomial vanishing on these points.**
   The space of such polynomials has dim 187000 (since $\dim L = 1$).

Toward Strassen’s asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

1. Use the secant variety $\sigma_{18}(V)$ with the parametrization: $\Phi_{18} : (u_i, v_i, w_i) \mapsto T = \sum^{18} u_i \otimes v_i \otimes w_i$
2. By **numerical implicitization**, we know that $\sigma_{18}(V)$ is codim 1 (a hypersurface) of degree ≥ 187000 (i.e., $|\sigma_{18}(V) \cap L| \geq 187000$) **(Hauenstein-Ikenmeyer-Landsberg 2013)**
3. **There is no degree 186999 homogeneous polynomial vanishing on these points.** The space of such polynomials has dim 187000 (since $\dim L = 1$).
4. For any $S \in \sigma_{18}(V)$, by sub-multiplicativity we have $\operatorname{rank}(S^{\otimes 186999}) \leq 18^{186999}$

Toward Strassen's asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

1. Use the secant variety $\sigma_{18}(V)$ with the parametrization: $\Phi_{18} : (u_i, v_i, w_i) \mapsto T = \sum^{18} u_i \otimes v_i \otimes w_i$
2. By **numerical implicitization**, we know that $\sigma_{18}(V)$ is codim 1 (a hypersurface) of degree ≥ 187000 (i.e., $|\sigma_{18}(V) \cap L| \geq 187000$) **(Hauenstein-Ikenmeyer-Landsberg 2013)**
3. **There is no degree 186999 homogeneous polynomial vanishing on these points.** The space of such polynomials has dim 187000 (since $\dim L = 1$).
4. For any $S \in \sigma_{18}(V)$, by sub-multiplicativity we have $\operatorname{rank}(S^{\otimes 186999}) \leq 18^{186999}$
5. The 187000 points in $\sigma_{18}(V) \cap L$ form a basis S_i of the space of degree 186999 polynomials. Hence,

$$\operatorname{rank}(T^{\otimes 186999}) = \operatorname{rank}(\sum c_i S_i) \leq 18^{186999} \cdot 187000$$

Toward Strassen's asymptotic rank conjecture

Case study $V = \mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$

Generic border rank is 19 (the standard upper bound). We want to show that the asymptotic rank is less than 19.

1. Use the secant variety $\sigma_{18}(V)$ with the parametrization: $\Phi_{18} : (u_i, v_i, w_i) \mapsto T = \sum^{18} u_i \otimes v_i \otimes w_i$
2. By **numerical implicitization**, we know that $\sigma_{18}(V)$ is codim 1 (a hypersurface) of degree ≥ 187000 (i.e., $|\sigma_{18}(V) \cap L| \geq 187000$) **(Hauenstein-Ikenmeyer-Landsberg 2013)**
3. **There is no degree 186999 homogeneous polynomial vanishing on these points.** The space of such polynomials has dim 187000 (since $\dim L = 1$).
4. For any $S \in \sigma_{18}(V)$, by sub-multiplicativity we have $\operatorname{rank}(S^{\otimes 186999}) \leq 18^{186999}$
5. The 187000 points in $\sigma_{18}(V) \cap L$ form a basis S_i of the space of degree 186999 polynomials. Hence,
$$\operatorname{rank}(T^{\otimes 186999}) = \operatorname{rank}(\sum c_i S_i) \leq 18^{186999} \cdot 187000$$
6. New bound: $$\tilde{R}(T) \leq \left(\operatorname{rank}(T^{\otimes 186999})\right)^{\frac{1}{186999}} \leq 18 \cdot 187000^{\frac{1}{186999}} < 18.001169$$

Toward Strassen's asymptotic rank conjecture

Geometric framework

Thm (Kaski-Michałek 2025) Let L be a fixed subspace of $V = \mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$, and $Y \subset L$ be a subset with the property that for all $T \in V$, the asymptotic rank is at most r. Suppose that there is no homogeneous polynomial on L of degree q that vanishes on Y. Then, every tensor in L has an asymptotic rank at most

$$r\binom{\dim L - 1 + q}{\dim L - 1}^{\frac{1}{q}}$$

Toward Strassen's asymptotic rank conjecture

Geometric framework

$Y = \sigma_{18}(V) \cap L$

<u>Thm</u> (Kaski-Michałek 2025) Let L be a fixed subspace of $V = \mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$, and $Y \subset L$ be a subset with the property that for all $T \in V$, the asymptotic rank is at most r. Suppose that there is no homogeneous polynomial on L of degree q that vanishes on Y. Then, every tensor in L has an asymptotic rank at most

186999 18

$$r\binom{\dim L - 1 + q}{\dim L - 1}^{\frac{1}{q}}$$

Toward Strassen's asymptotic rank conjecture

Geometric framework

Thm (Kaski-Michałek 2025) Let L be a fixed subspace of $V = \mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$, and $Y \subset L$ be a subset with the property that for all $T \in V$, the asymptotic rank is at most r. Suppose that there is no homogeneous polynomial on L of degree q that vanishes on Y. Then, every tensor in L has an asymptotic rank at most

$$r \binom{\dim L - 1 + q}{\dim L - 1}^{\frac{1}{q}}$$

Toward Strassen's asymptotic rank conjecture

Geometric framework

Thm (Kaski-Michałek 2025) Let L be a fixed subspace of $V = \mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$, and $Y \subset L$ be a subset with the property that for all $T \in V$, the asymptotic rank is at most r. Suppose that there is no homogeneous polynomial on L of degree q that vanishes on Y. Then, every tensor in L has an asymptotic rank at most

$$r\binom{\dim L - 1 + q}{\dim L - 1}^{\frac{1}{q}}$$

Goal: Systematically find all tensor spaces with strictly improved asymptotic rank bounds.

Toward Strassen's asymptotic rank conjecture

Geometric framework

Thm (Kaski-Michałek 2025) Let L be a fixed subspace of $V = \mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$, and $Y \subset L$ be a subset with the property that for all $T \in V$, the asymptotic rank is at most r. Suppose that there is no homogeneous polynomial on L of degree q that vanishes on Y. Then, every tensor in L has an asymptotic rank at most

$$r \binom{\dim L - 1 + q}{\dim L - 1}^{\frac{1}{q}}$$

Goal: Systematically find all tensor spaces with strictly improved asymptotic rank bounds.

Core question: How large is the degree q such that no polynomial vanishes on Y?

Toward Strassen's asymptotic rank conjecture

Geometric framework

Thm (Kaski-Michałek 2025) Let L be a fixed subspace of $V = \mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$, and $Y \subset L$ be a subset with the property that for all $T \in V$, the asymptotic rank is at most r. Suppose that there is no homogeneous polynomial on L of degree q that vanishes on Y. Then, every tensor in L has an asymptotic rank at most

$$r\binom{\dim L - 1 + q}{\dim L - 1}^{\frac{1}{q}}$$

Goal: Systematically find all tensor spaces with strictly improved asymptotic rank bounds.

Core question: How large is the degree q such that no polynomial vanishes on Y?

- For codim 1, we have $q = \deg(\sigma_r(V)) - 1$

Computational remark

Formulating the polynomial system

Computational remark

Formulating the polynomial system

We exploit the multilinear structure to improve the efficiency of homotopy continuation.

The parametrized tensor

$$\sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

where $u_i \in \mathbb{C}^{a-1}, v_i \in \mathbb{C}^{b-1}, w_i \in \mathbb{C}^{c}$

$=$

A generic linear slice

$$L = \{At + B \mid t \in \mathbb{C}^{\ell}\}$$

where $\ell = \operatorname{codim}(\sigma_r(V))$

Computational remark

Formulating the polynomial system

We exploit the multilinear structure to improve the efficiency of homotopy continuation.

The parametrized tensor

$$\sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

where $u_i \in \mathbb{C}^{a-1}, v_i \in \mathbb{C}^{b-1}, w_i \in \mathbb{C}^{c}$

$=$

A generic linear slice

$$L = \{At + B \mid t \in \mathbb{C}^{\ell}\}$$

where $\ell = \mathrm{codim}(\sigma_r(V))$

The system to solve:

$$\sum_{i=1}^{r} u_i \otimes v_i \otimes w_i - (At + B) = 0$$

$r(a+b+c-2)+\ell$ variables (u_i, v_i, w_i, t) and parameters $A \in \mathbb{C}^{abc\times\ell}, B \in \mathbb{C}^{abc}$.
(with additional generic linear slices if necessary)

Results
codimension 1 cases

Results

codimension 1 cases

Scope: Systematic search for all concise cases $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ of $r \leq 20$ where $\sigma_r(V)$ has codim ≤ 3.

Results
codimension 1 cases

Scope: Systematic search for all concise cases $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ of $r \leq 20$ where $\sigma_r(V)$ has codim ≤ 3.

Approach: Sample many points from $\sigma_r(V)$ (using HomotopyContinuation.jl) to find a new asymptotic rank bound.

Results

codimension 1 cases

Scope: Systematic search for all concise cases $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ of $r \leq 20$ where $\sigma_r(V)$ has codim ≤ 3.

Approach: Sample many points from $\sigma_r(V)$ (using HomotopyContinuation.jl) to find a new asymptotic rank bound.

Hardware: All experiments were performed on a HPC cluster with 16 Genoa-generation AMD CPU cores and 64 GB of RAM, with a maximum wall time of 72 hours.

Results

codimension 1 cases

Scope: Systematic search for all concise cases $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ of $r \leq 20$ where $\sigma_r(V)$ has codim ≤ 3.

Approach: Sample many points from $\sigma_r(V)$ (using HomotopyContinuation.jl) to find a new asymptotic rank bound.

Hardware: All experiments were performed on a HPC cluster with 16 Genoa-generation AMD CPU cores and 64 GB of RAM, with a maximum wall time of 72 hours.

Codimension 1

(a,b,c)	r	**Non-defective?**	$\deg \sigma_r(V)$	**New bound**
$(3,2n+1,2n+1)$	$3n+1$	No	$6n+3$	N/A
$(3,5,7)$	9	Yes	105 **(HIL 2013)**	< 8.366128
$(4,7,14)$	17	Yes	≥ 1229	< 17.098769
$(6,6,9)$	17	Yes	≥ 3601	< 17.038715
$(7,7,7)$	18	Yes	≥ 187000 **(HIL 2013)**	< 18.001169
$(5,8,10)$	19	Yes	≥ 3638	< 19.042882

Note: *N/A indicates the degree is insufficient to improve upon the generic bound.*

Non-defective indicates that $\operatorname{codim}(\sigma_r(V)) = abc - r(a+b+c-2)$

Results

codimension ≥ 2 cases

Results

codimension ≥ 2 cases

Unlike codimension 1 cases, when $\mathrm{codim}(\sigma_r(V)) \geq 2$, $\deg(\sigma_r(V))$ does not induce the degree of non-vanishing polynomial.

Results

codimension ≥ 2 cases

Unlike codimension 1 cases, when $\mathrm{codim}(\sigma_r(V)) \geq 2$, $\deg(\sigma_r(V))$ does not induce the degree of non-vanishing polynomial.

Approach: Sample many, many, many, many points (say D) from $\sigma_r(V)$ and do the **interpolation**.

Results

codimension ≥ 2 cases

Unlike codimension 1 cases, when $\mathrm{codim}(\sigma_r(V)) \geq 2$, $\deg(\sigma_r(V))$ does not induce the degree of non-vanishing polynomial.

Approach: Sample **many, many, many, many** points (say D) from $\sigma_r(V)$ and do the **interpolation**.

$$M_q = \begin{bmatrix} x_1^q & x_1^{q-1}y_1 & x_1^{q-1}z_1 & x_1^{q-2}y_1^2 & x_1^{q-2}y_1z_1 & x_1^{q-2}z_1^2 & \cdots & z_1^q \\ x_2^q & x_2^{q-1}y_2 & x_2^{q-1}z_2 & x_2^{q-2}y_2^2 & x_2^{q-2}y_2z_2 & x_2^{q-2}z_2^2 & \cdots & z_2^q \\ & & & \vdots & & & & \\ x_D^q & x_D^{q-1}y_D & x_D^{q-1}z_D & x_D^{q-2}y_D^2 & x_D^{q-2}y_Dz_D & x_D^{q-2}z_D^2 & \cdots & z_D^q \end{bmatrix}$$

Goal: If $\mathrm{rank}(M_q)$ is full rank, then no degree q polynomial vanishes on $\sigma_r(V) \cap L$

Results

codimension ≥ 2 cases

Unlike codimension 1 cases, when $\mathrm{codim}(\sigma_r(V)) \geq 2$, $\deg(\sigma_r(V))$ does not induce the degree of non-vanishing polynomial.

Approach: Sample many, many, many, many points (say D) from $\sigma_r(V)$ and do the **interpolation**.

$$M_q = \begin{bmatrix} x_1^q & x_1^{q-1}y_1 & x_1^{q-1}z_1 & x_1^{q-2}y_1^2 & x_1^{q-2}y_1z_1 & x_1^{q-2}z_1^2 & \cdots & z_1^q \\ x_2^q & x_2^{q-1}y_2 & x_2^{q-1}z_2 & x_2^{q-2}y_2^2 & x_2^{q-2}y_2z_2 & x_2^{q-2}z_2^2 & \cdots & z_2^q \\ & & & \vdots & & & & \\ x_D^q & x_D^{q-1}y_D & x_D^{q-1}z_D & x_D^{q-2}y_D^2 & x_D^{q-2}y_Dz_D & x_D^{q-2}z_D^2 & \cdots & z_D^q \end{bmatrix}$$

Goal: If $\mathrm{rank}(M_q)$ is full rank, then no degree q polynomial vanishes on $\sigma_r(V) \cap L$

Reality: As q grows, the matrix becomes numerically ill-conditioned. (e.g., $0.9^{100} \approx 0.000027$, $1.1^{100} \approx 13780$)

Results

codimension ≥ 2 cases

Unlike codimension 1 cases, when $\mathrm{codim}(\sigma_r(V)) \geq 2$, $\deg(\sigma_r(V))$ does not induce the degree of non-vanishing polynomial.

Approach: Sample **many, many, many, many** points (say D) from $\sigma_r(V)$ and do the **interpolation**.

$$M_q = \begin{bmatrix} x_1^q & x_1^{q-1}y_1 & x_1^{q-1}z_1 & x_1^{q-2}y_1^2 & x_1^{q-2}y_1z_1 & x_1^{q-2}z_1^2 & \cdots & z_1^q \\ x_2^q & x_2^{q-1}y_2 & x_2^{q-1}z_2 & x_2^{q-2}y_2^2 & x_2^{q-2}y_2z_2 & x_2^{q-2}z_2^2 & \cdots & z_2^q \\ & & & \vdots & & & & \\ x_D^q & x_D^{q-1}y_D & x_D^{q-1}z_D & x_D^{q-2}y_D^2 & x_D^{q-2}y_Dz_D & x_D^{q-2}z_D^2 & \cdots & z_D^q \end{bmatrix}$$

Goal: If $\mathrm{rank}(M_q)$ is full rank, then no degree q polynomial vanishes on $\sigma_r(V) \cap L$

Reality: As q grows, the matrix becomes numerically ill-conditioned. (e.g., $0.9^{100} \approx 0.000027$, $1.1^{100} \approx 13780$)

Spoiler alert: We observed **no** improved bounds from $\mathrm{codim} \geq 2$ cases.

Results

codimension ≥ 2 cases

Codimension 2

(a, b, c)	r	**Non-defective?**	$\deg \sigma_r(V)$
(2,2,3)	2	Yes	6
(2,3,5)	4	No	15
(2,4,7)	6	No	28
(3,3,8)	7	No	≥ 36
(2,5,9)	8	No	≥ 45
(4,4,8)	9	Yes	≥ 30005
(2,6,11)	10	No	≥ 65
(3,6,9)	10	Yes	≥ 78589

Codimension 3

(a, b, c)	r	**Non-defective?**	$\deg \sigma_r(V)$
(2,3,4)	3	Yes	20
(2,4,6)	5	No	56
(3,3,7)	6	Yes	90
(2,5,8)	7	No	≥ 120
(4,4,5)	7	Yes	44000 **(HIL 2013)**
(2,6,10)	9	No	≥ 220
(4,5,6)	9	Yes	≥ 33634
(3,4,10)	9	No	≥ 219

Results

codimension ≥ 2 cases

Codimension 2

(a,b,c)	r	Non-defective?	$\deg \sigma_r(V)$
(2,2,3)	2	Yes	6
(2,3,5)	4	No	15
(2,4,7)	6	No	28
(3,3,8)	7	No	≥ 36
(2,5,9)	8	No	≥ 45
(4,4,8)	9	Yes	≥ 30005
(2,6,11)	10	No	≥ 65
(3,6,9)	10	Yes	≥ 78589

Codimension 3

(a,b,c)	r	Non-defective?	$\deg \sigma_r(V)$
(2,3,4)	3	Yes	20
(2,4,6)	5	No	56
(3,3,7)	6	Yes	90
(2,5,8)	7	No	≥ 120
(4,4,5)	7	Yes	44000 **(HIL 2013)**
(2,6,10)	9	No	≥ 220
(4,5,6)	9	Yes	≥ 33634
(3,4,10)	9	No	≥ 219

- The smallest promising codim 2 case is $\sigma_9(4,4,8)$. We need $q \geq 76$ to improve the bound. This requires checking the rank of a matrix of size at least 3003×3003. Numerical instability prevented reliable rank verification.

Results

codimension ≥ 2 cases

Codimension 2

(a, b, c)	r	**Non-defective?**	$\deg \sigma_r(V)$
(2,2,3)	2	Yes	6
(2,3,5)	4	No	15
(2,4,7)	6	No	28
(3,3,8)	7	No	≥ 36
(2,5,9)	8	No	≥ 45
(4,4,8)	9	Yes	≥ 30005
(2,6,11)	10	No	≥ 65
(3,6,9)	10	Yes	≥ 78589

Codimension 3

(a, b, c)	r	**Non-defective?**	$\deg \sigma_r(V)$
(2,3,4)	3	Yes	20
(2,4,6)	5	No	56
(3,3,7)	6	Yes	90
(2,5,8)	7	No	≥ 120
(4,4,5)	7	Yes	44000 **(HIL 2013)**
(2,6,10)	9	No	≥ 220
(4,5,6)	9	Yes	≥ 33634
(3,4,10)	9	No	≥ 219

- The smallest promising codim 2 case is $\sigma_9(4,4,8)$. We need $q \geq 76$ to improve the bound. This requires checking the rank of a matrix of size at least 3003×3003. Numerical instability prevented reliable rank verification.
- For codim 4 cases, the matrix size explodes to > 1800000

Results
codimension ≥ 2 cases

Codimension 2

(a,b,c)	r	**Non-defective?**	$\deg \sigma_r(V)$
(2,2,3)	2	Yes	6
(2,3,5)	4	No	15
(2,4,7)	6	No	28
(3,3,8)	7	No	≥ 36
(2,5,9)	8	No	≥ 45
(4,4,8)	9	Yes	≥ 30005
(2,6,11)	10	No	≥ 65
(3,6,9)	10	Yes	≥ 78589

Codimension 3

(a,b,c)	r	**Non-defective?**	$\deg \sigma_r(V)$
(2,3,4)	3	Yes	20
(2,4,6)	5	No	56
(3,3,7)	6	Yes	90
(2,5,8)	7	No	≥ 120
(4,4,5)	7	Yes	44000 **(HIL 2013)**
(2,6,10)	9	No	≥ 220
(4,5,6)	9	Yes	≥ 33634
(3,4,10)	9	No	≥ 219

- The smallest promising codim 2 case is $\sigma_9(4,4,8)$. We need $q \geq 76$ to improve the bound. This requires checking the rank of a matrix of size at least 3003×3003. Numerical instability prevented reliable rank verification.
- For codim 4 cases, the matrix size explodes to > 1800000

Conjecture For large enough non-defective cases of codimension 1, we have $\tilde{R} < (\text{generic border rank})$

Final recap

Core question again

Final recap
Core question again

Core question:

How large is the degree q such that no degree q polynomial vanishes on $Y = \sigma_r(V) \cap L$?

Final recap

Core question again

Core question:

How large is the degree q such that no degree q polynomial vanishes on $Y = \sigma_r(V) \cap L$?

- For codim 1, we have $q = \deg(\sigma_r(V)) - 1$.
- For codim ≥ 2, interpolation is the only (ineffective) way.

Final recap

Core question again

Core question:

How large is the degree q such that no degree q polynomial vanishes on $Y = \sigma_r(V) \cap L$?

- For codim 1, we have $q = \deg(\sigma_r(V)) - 1$.
- For codim ≥ 2, interpolation is the only (ineffective) way.

Thank you for your attention!